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PARTITION IDENTITIES INVOLVING GAPS AND WEIGHTS

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Abstract. We obtain interesting new identities connecting the famous partition functions of Euler, Gauss, Lebesgue, Rogers–Ramanujan and others by attaching weights to the gaps between parts. The weights are in general multiplicative. Some identities involve powers of 2 as weights and yield combinatorial information about some remarkable partition congruences modulo powers of 2.

1. Introduction

Four of the most fundamental and extensively studied partition functions are:

(i) $p(n)$ = the number of unrestricted partitions of n,
(ii) $Q(n)$ = the number of partitions of n into distinct parts,
(iii) $g(n)$ = the number of partitions of n in which the even parts do not repeat, and
(iv) $\rho(n)$ = the number of partitions of n with difference ≥ 2 between parts.

In laying the foundations of the theory of partitions, Euler established several generating function identities involving $p(n)$ and $Q(n)$. The function $g(n)$ appears in a famous identity due to Lebesgue and Gauss. Finally, $\rho(n)$ is the function appearing in the celebrated (first) Rogers–Ramanujan identity. Andrews' encyclopedia [5] contains a wealth of information about these and related partition functions.

Our purpose here is to provide some new relationships for these classical partition functions by attaching weights which are defined multiplicatively using gaps between parts. Such weighted identities have many applications, a few of which are briefly discussed here. Certain applications of great interest are discussed in detail elsewhere (see [1], [2]). For instance, the most striking application (see [1]) is a combinatorial correspondence for a deep partition theorem of Göllnitz [11], which is obtained by suitably extending the weighted identity in Theorem 15 of §7 and interpreting this extension combinatorially.

To understand these weighted identities in a proper setting, consider the following general problem: Given a set S of partitions, let $P_S(n)$ denote the number of partitions π of n with $\pi \in S$. Suppose $S \subseteq T$. Our problem is to determine weights

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$w_S(\pi) \geq 1$ in a natural way such that for all n

$$P_T(n) = \sum_{\substack{\sigma(\pi)=n \\ \pi \in S}} w_S(\pi). \tag{1.1}$$

(Here and in what follows, $\sigma(\pi)$ is the sum of the parts of π; for further notation see §2.) Note that determining weights $w_T(\pi)$ such that

$$P_S(n) = \sum_{\substack{\sigma(\pi)=n \\ \pi \in T}} w_T(\pi) \tag{1.2}$$

is trivial, because $S \subseteq T$ implies that

$$w_T(\pi) = \begin{cases} 1, & \text{if } \pi \in S, \\ 0, & \text{if } \pi \notin S, \end{cases} \tag{1.3}$$

is a valid choice. One may view (1.1) as some sort of an "inversion" of (1.2), but no such general inversion formula is known. For most choices of S and T, finding the weights $w_S(\pi)$ is quite difficult. Our primary purpose here is to produce such weights when $P_S(n)$ and $P_T(n)$ are the classical functions listed above.

After introducing notation and partition preliminaries in §2, a weighted identity connecting $P_S(n) = \rho(n)$ and $P_T(n) = p(n)$ is derived in §3 (see Theorem 1). The Rogers–Ramanujan partition function has another interpretation; namely, $\rho(n)$ is the number of partitions of n where each part is $\geq$ the number of parts. This second interpretation yields the identity in Theorem 2, §3, which has the advantage that it extends to the case where $P_T(n) = p(n)$ and $P_S(n)$ is the number of partitions of n into fewer than k Durfee squares (see Theorem 8, §4).

Next, by utilizing a variation in a combinatorial technique due to Bressoud [9], a weighted identity connecting $Q(n)$ and $\rho(n)$ in derived in §5.

In terms of applications, the results of §§6 and 7 are the most significant. Section 6 discusses a result proved by Alladi and Gordon [4] connecting $Q(n)$ and $g(n)$ via the identity

$$g(n) = \sum_k Q(n,k)2^k \tag{1.4}$$

and its refinement. Here $Q(n,k)$ is the number of partitions $b_1 + b_2 + \cdots + b_\nu$ counted by $Q(n)$ with the added restriction that there are exactly k gaps $b_i - b_{i+1}$ which are ≥ 2 with the convention that $b_{\nu+1} = 0$. One reason (1.4) is of interest is because it indicates that $g(n)$ is almost always a multiple of any given power of 2. In §7, a similar result is derived for $Q(n)$, a result I had proved in 1990, namely,

$$Q(n) = \sum_k g_3(n,k)2^k. \tag{1.5}$$

For a precise definition of $g_3(n,k)$, see §7. I had proved (1.5) by a combinatorial study of an identity of Sylvester [16] in his classic paper of 1882, and this proof is given in §7. Identity (1.5) suggests the remarkable result that for each positive integer k, the congruence

$$Q(n) \equiv 0 (\text{mod } 2^k) \tag{1.6}$$

holds for almost all values of n. A proof of (1.6) and an extension of it to p-regular partitions has been given by Gordon and Ono [13] making use of the theory of modular forms. Naturally it is of interest to give a combinatorial proof of the validity of (1.6) for almost all n, and (1.5) appears to be the key to the combinatorial understanding of (1.6). Similarly, (1.4) is the key to the combinatorial study of the related congruence

$$g(n) \equiv 0 (\mathrm{mod}\ 2^k). \tag{1.7}$$

In a forthcoming paper [2] we utilize (1.4) and (1.5) to show combinatorially that for small k, (1.6) and (1.7) are valid almost always, and to determine the exceptional sets of zero density where the congruences fail.

Recently, Bowman [8] has obtained many interesting results for various classical partition functions by inserting 'ones' in gaps. Bowman's notions are different from ours and so are his results, which are obtained by a study of continued fractions. But there appear to be connections between his approach and ours which may be worth a deeper study.

2. Partition preliminaries

Given any complex number a and a positive integer n, set

$$(a)_n = (a;q)_n = \prod_{n=0}^{n-1}(1-aq^j),$$

and for $|q|<1$,

$$(a)_\infty = \lim_{n\to\infty}(a)_n = \prod_{j=0}^{\infty}(1-aq^j).$$

We use the symbol π to denote a partition as well as its Ferrers graph. The conjugate of the Ferrers graph of π is denoted by π^*.

The Ferrers graph of any partition π contains a largest square $D_1(\pi) = D_1$ of nodes starting from the left hand top corner as is shown in Figure 1. This is the Durfee square of the Ferrers graph. The nodes to the right of $D_1(\pi)$ constitute a partition denoted by π_r. Similarly, the nodes below $D_1(\pi)$ constitute another partition denoted by π_b.

Given a partition π: $b_1+b_2+\cdots+b_\nu$ with $b_1\ge b_2\ge\cdots\ge b_\nu>0$, we let $\sigma(\pi)$ denote the sum of the parts of π and $\nu(\pi)=\nu$, the number of parts of π. Also $\nu^*(\pi)$ is the number of different parts of π (not counting repetitions) and $\lambda(\pi)=b_1$,

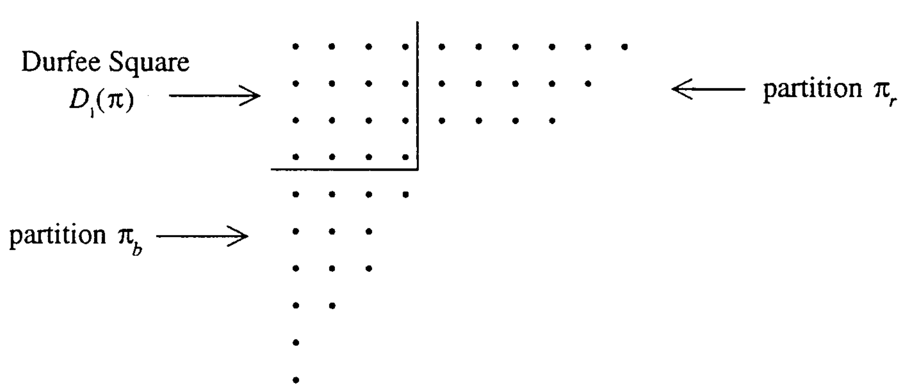

Figure 1

the largest part of π. Sometimes we need to keep track of parts of a certain type, and this will be indicated by a subscript.

Given two partitions π_1 and π_2, by $\pi_1+\pi_2$ we mean the partition whose parts are obtained by adding the number of nodes in the corresponding rows of the Ferrers graphs of π_1 and π_2.

Since each partition π may be viewed as a multi-set whose elements are positive integers arranged in decreasing order, we define $\pi_1 \cup \pi_2$ to be the partition obtained by considering the set theoretic union of π_1 and π_2.

For each integer $j \geq 1$, the function

$$1+\frac{bzq^j}{1-zq^j}=\frac{1+z(1-b)q^j}{1-zq^j} \tag{2.1}$$

is the generating function for partitions π using only the part j, where each non–empty partition is counted with weight $bz^{\nu(\pi)}$. Thus

$$\frac{(-z(1-b)q)_n}{(zq)_n}=\sum_{\lambda(\pi)\leq n} b^{\nu^*(\pi)}z^{\nu(\pi)}q^{\sigma(\pi)} \tag{2.2}$$

is the generating function for partitions π with largest part $\leq n$, with the power of b keeping track of the number of different parts and the power of z counting the total number of parts.

For a partition π whose parts are $b_1 \geq b_2 \geq \cdots \geq b_\nu > 0$, the gaps in the partition are the differences $b_i - b_{i+1}$ for $i = 1, 2, \ldots, \nu$, where $b_{\nu+1}$ will be defined suitably based on the type of partition being discussed. For instance, in §6 we take $b_{\nu+1} = 0$, whereas in §7, $b_{\nu+1} = -1$.

3. Unrestricted and Rogers–Ramanujan partitions

One of the fundamental identities that Euler proved for the number of unrestricted partitions of n is

$$\sum_{n=0}^{\infty} p(n)q^n=\sum_{\nu=0}^{\infty}\frac{q^{\nu^2}}{(q)_\nu^2}=\frac{1}{(q)_\infty}. \tag{3.1}$$

Identity (3.1) is best understood by appeal to Durfee squares as in [14].

The celebrated first Rogers–Ramanujan identity is

$$\sum_{\nu=0}^{\infty}\frac{q^{\nu^2}}{(q)_\nu}=\frac{1}{(q;q^5)_\infty(q^4;q^5)_\infty}. \tag{3.2}$$

The combinatorial interpretation of the product on the right of (3.2) is clear: it is the generating function of partitions into parts $\equiv \pm 1 (\mathrm{mod}\ 5)$. One combinatorial interpretation for the left hand side is that it is the generating function of $\rho(n)$, the number of partitions of n into parts differing by ≥ 2. More specifically,

$$\frac{q^{\nu^2}}{(q)_\nu} \tag{3.3}$$

is the generating function for partitions into ν parts differing by ≥ 2. Another interpretation for the term in (3.3) is that it is the generating function for partitions with a $\nu \times \nu$ Durfee square such that there are no nodes below the Durfee square as shown in Figure 2. That is, the partition π_b (see Figure 1) does not exist here. We call such partitions as *primary* partitions.

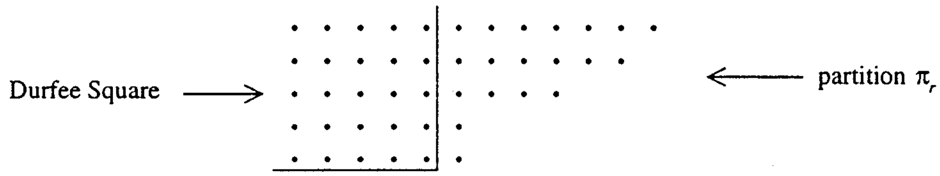

FIGURE 2

For primary partitions, the number of nodes in each row is at least as large as the number of rows. Thus $\rho(n)$ is also the number of partitions of n into parts each at least as large as the number of parts. To get a bijection between partitions π_1 of n into parts $b_1 \geq b_2 \geq \cdots \geq b_\nu$ with $b_\nu \geq \nu$ and partitions $\tilde{\pi}$ of n into ν parts $h_1 + h_2 + \cdots + h_\nu$ differing by ≥ 2, just count the number of nodes along hooks in the Ferrers graphs of the primary partitions. For the graph in Figure 2, the partition with minimal difference 2 that it generates is $16 + 13 + 9 + 4 + 2$ by counting nodes along hooks. In view of this bijection, and for later use, it is convenient to introduce the notation $\rho(\pi) = \tilde{\pi}$ to mean the partition obtained by counting nodes along hooks of the Ferrers graph of π.

Consider the decomposition

$$\frac{q^{\nu^2}}{(q)_\nu^2} = \frac{q^{\nu^2}}{(q)_\nu} \cdot \frac{1}{(q)_\nu}. \tag{3.4}$$

To obtain a combinatorial interpretation of this decomposition, we define the *sliding operation* on Ferrers graphs.

Given a Ferrers graph π corresponding to a partition of n, by a sliding operation ψ on π we mean the removal of certain columns to the right of the Durfee square of π and the placement of these columns as rows below the Durfee square so as to form another Ferrers graph π'. Several sliding operations ψ can be performed on each partition π. We write $\pi' = \psi(\pi)$ if π' is obtained from π by a certain sliding operation. The following are invariants under the operation ψ.

(i) The total number of nodes in π and π' are the same, that is, $\sigma(\pi) = \sigma(\pi')$.
(ii) The partitions π and π' have the same Durfee square. That is, $D_1(\pi) = D_1(\pi')$.
(iii) The sum of the number of parts and the largest part is invariant. That is,

$$\nu(\pi) + \lambda(\pi) = \nu(\pi') + \lambda(\pi').$$

(iv) Most importantly, the partition obtained by counting nodes along the hooks of π and π' will be the same. That is,

$$\rho(\pi) = \rho(\pi').$$

In the partition displayed in Figure 3, up to $d_3 = 3$ columns of length 3 could be moved from the right of the Durfee square and placed below the square. More generally, given a primary partition π as in Figure 2, with parts $b_1, b_2, \ldots, b_\nu$, let $h_1, h_2, \ldots, h_\nu$ be the parts of $\tilde{\pi} = \rho(\pi)$ satisfying $h_i - h_{i+1} \geq 2$, for $1 \leq i \leq \nu - 1$. Now write $h_i - h_{i+1} = 2 + d_i$. Notice that up to d_i vertical columns of length i could be removed from the right of the $D(\pi)$ and placed below the Durfee square as rows. Thus the number of choices of moving columns of length i is $d_i + 1 = h_i - h_{i+1} - 1$, for $1 \leq i \leq \nu - 1$. For $i = \nu$, up to $h_\nu - 1$ columns of length ν could be moved.

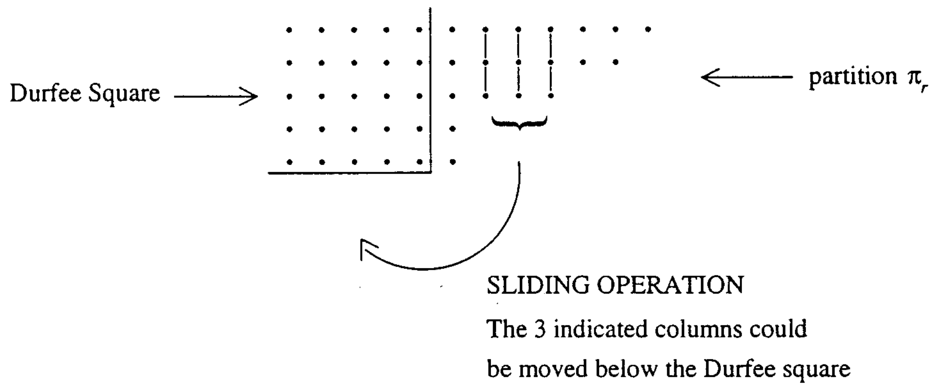

FIGURE 3

Thus the number of choices here is h_ν. So this leads us to define a weight function for Rogers–Ramanujan partitions $\tilde{\pi}$ as follows:

$$w_R(\tilde{\pi}) = \prod_{i=1}^{\nu-1}(h_i - h_{i+1} - 1) \cdot h_\nu. \tag{3.5}$$

In view of the invariant (iv) above, we see that under the sliding operation each Rogers–Ramanujan partition $\tilde{\pi}$ of n generates $w_R(\tilde{\pi})$ unrestricted partitions of n, and that all partitions of n can be generated in this manner. So we have proved

Theorem 1. *Let R denote the set of all partitions with minimal difference 2. Then*

$$p(n) = \sum_{\substack{\sigma(\tilde{\pi})=n \\ \tilde{\pi}\in R}} w_R(\tilde{\pi}).$$

If we define $h_{\nu+1} = -1$, then the weights in (3.5) could be cast in the more elegant form

$$w_R(\tilde{\pi}) = \prod_{i=1}^{\nu}(h_i - h_{i+1} - 1). \tag{3.6}$$

The connection between the parts b_i of a primary partition π and the parts h_i of the partition $\tilde{\pi} = \rho(\pi)$ is

$$b_i = h_i - (2(\nu - i) + 1), \quad \text{for} \quad 1 \le i \le \nu. \tag{3.7}$$

Now (3.7) implies that

$$b_i - b_{i+1} + 1 = h_i - h_{i+1} - 1, \quad \text{for} \quad 1 \le i \le \nu, \tag{3.8}$$

with the conventions $b_{\nu+1} = \nu$ and $h_{\nu+1} = -1$. Thus under the sliding operation each primary partition π with parts $b_1, b_2, \ldots, b_\nu$ generates $w_1(\pi)$ unrestricted partitions, where

$$w_1(\pi) = \prod_{i=1}^{\nu}(b_i - b_{i+1} + 1). \tag{3.9}$$

(Compare (3.9) with (3.6).) This leads to Theorem 2 below, which may be viewed as a companion to Theorem 1.

Theorem 2. *Let D_1 denote the set of all partitions each of whose parts is at least as large as the number of parts. Then*

$$p(n) = \sum_{\substack{\sigma(\pi)=n \\ \pi\in D_1}} w_1(\pi).$$

The invariants (ii) and (iii) under the sliding operation lead to refinements of Theorem 1 and 2 as stated below.

Theorem 3. *Let $p(n;\nu)$ denote the number of partitions of n having a $\nu\times\nu$ Durfee square.*

Let R_ν and $D_{1,\nu}$ denote the subsets of R and D_1 consisting of partitions having exactly ν parts. Then

$$p(n;\nu) = \sum_{\substack{\sigma(\tilde\pi)=n \\ \tilde\pi\in R_\nu}} w_R(\tilde\pi) = \sum_{\substack{\sigma(\pi)=n \\ \pi\in D_{1,\nu}}} w_1(\pi).$$

Theorem 4. *Let $p_m(n)$ denote the number of partitions π of n such that $\lambda(\pi) + \nu(\pi) = m+1$.*

Let $R(m)$ denote the subset of R consisting of partitions $\tilde\pi$ for which $\lambda(\tilde\pi) = m$.

Let $D_1(m)$ denote the subset of D_1 consisting of partitions π such that $\lambda(\pi) + \nu(\pi) = m+1$. Then

$$p_m(n) = \sum_{\substack{\sigma(\tilde\pi)=n \\ \tilde\pi\in R(m)}} w_R(\tilde\pi) = \sum_{\substack{\sigma(\pi)=n \\ \pi\in D_1(m)}} w_1(\pi).$$

4. Partitions into $< k$ Durfee squares

The partition theoretic form of the first Rogers–Ramanujan identity (3.2) is

Theorem 5. *The number of partitions $\rho(n)$ of n into parts with minimal difference 2 is equal to the number of partitions of n into parts $\equiv \pm1 (mod\ 5)$.*

As was noticed earlier, $\rho(n)$ is also the number of partitions of n into parts each at least as large as the number of parts.

The second Rogers–Ramanujan identity is

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q)_n} = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}. \tag{4.1}$$

This has the following combinatorial interpretation.

Theorem 5′. *The number of partitions $\rho'(n)$ into parts ≥ 2 and with minimal difference 2 equals the number of partitions of n into parts $\equiv \pm2 (mod\ 5)$.*

In this case $\rho'(n)$ is also equal to the number of partitions of n into parts each of which is greater than the number of parts.

In the 1960's, Gordon [12] obtained the following significant extension of Theorems 5 and 5′ to an odd modulus $2k+1$.

Theorem 6. *Let i and k be integers satisfying $1 \leq i \leq k$ and $k > 1$. Then the number $\rho_{k,i}(n)$ of partitions of n in the form $\pi = h_1 + h_2 + \cdots + h_\nu$ such that $h_j - h_{j+k-1} \geq 2$ and with π having fewer than i ones equals the number of partitions of n into parts $\not\equiv 0, \pm i (mod\ 2k+1)$.*

Theorem 5 is the case $k = 2$, $i = 2$ and Theorem 5′ the case $k = 2, i = 1$ in Theorem 6.

Extending the interpretation of $\rho(n)$ in terms of Durfee squares, Andrews [6] showed

Theorem 7. *The function $\rho_{k,k}(n)$ is equal to the number of partitions of n whose Ferrers graphs have fewer than k successive Durfee squares.*

Successive Durfee squares in a partition $\pi = \pi_1$ can be defined as follows. We already know what the Durfee square of π is. We will refer to this as the first Durfee square of $\pi = \pi_1$ and denote this by $D_1(\pi) = D_1(\pi_1)$. The portion of the Ferrers graph below $D_1(\pi)$ can be considered to be another Ferrers graph π_2 which has its own Durfee square $D_1(\pi_2)$. We call this the second Durfee square of π_1 and denote it by $D_2(\pi_1)$. The portion of the Ferrers graph below $D_2(\pi_1)$ is another Ferrers graph π_3 with its own Durfee square $D_1(\pi_3) = D_3(\pi_1)$, the third Durfee square of π_1, and so on. Thus each Ferrers graph π_1 has its own sequence of successive Durfee squares $D_k(\pi_1)$. In this sense, $\rho(n) = \rho_{2,2}(n)$ is equal to the number of partitions of n whose Ferrers graphs have fewer than two Durfee squares. Partitions having at least two Durfee squares may be called non-Rogers-Ramanujan partitions as in [10].

A natural question to ask at this point is whether there is a weighted identity connecting $p(n)$ with the partitions counted by $\rho_{k,k}(n)$. The answer to this is in the affirmative, as shown below.

A partition into k successive Durfee squares can be written in the form

$$\begin{aligned}\pi =& b_1 + b_2 + \cdots + b_{\nu_1}\\ &+ b_{\nu_1+1} + b_{\nu_1+2} + \cdots + b_{\nu_1+\nu_2}\\ &+ b_{\nu_1+\nu_2+1} + b_{\nu_1+\nu_2+2} + \cdots + b_{\nu_1+\nu_2+\nu_3}\\ &\ddots\\ &+ b_{\nu_1+\nu_2+\cdots+\nu_{k-1}+1} + b_{\nu_1+\nu_2+\cdots+\nu_{k-1}+2} + b_{\nu_1+\nu_2+\cdots+\nu_k},\end{aligned}$$

with the conditions

$$b_{\nu_1} \geq \nu_1,\ b_{\nu_1+\nu_2} \geq \nu_2, \ldots,\ b_{\nu_1+\nu_2+\cdots+\nu_k} \geq \nu_k.$$

Here $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_k$ denote the sizes of the successive Durfee squares of π.

Consider the parts $b_1, b_2, \ldots, b_{\nu_k}$. Observe that if $1 \leq i < \nu_k$, then up to $b_i - b_{i+1}$ columns of length i lying to the right of $D_1(\pi)$ could be extracted and placed horizontally below $D_k(\pi)$ to form a partition with $\geq k$ Durfee squares. For $i = \nu_k$, up to $b_{\nu_k} - b_{\nu_k+1}$ columns of length ν_k could be extracted if $\nu_k < \nu_1$ or up to $b_{\nu_k} - \nu_k$ columns of length ν_k could be extracted if $\nu_k = \nu_1$. These extracted columns are to be placed horizontally below $D_k(\pi)$ to form a Ferrers graph with $\geq k$ Durfee squares. Now define the weights $w_k(\pi)$ as follows:

$$w_k(\pi) = \begin{cases} 0, & \text{if } \pi \text{ has} > k \text{ Durfee squares,}\\ 1, & \text{if } \pi \text{ has} < k \text{ Durfee squares,}\end{cases} \tag{4.2}$$

and for partitions π with exactly k Durfee squares

$$w_k(\pi) = \prod_{i=1}^{\nu_k}(b_i - b_{i+1} + 1),\ \text{if } \nu_k < \nu_1, \tag{4.3}$$

or

(4.4)
$$w_k(\pi) = (b_1 - b_2 + 1)(b_2 - b_3 + 1)\dots(b_{\nu_k - 1} - b_{\nu_k} + 1)(b_{\nu_k} - \nu_k + 1), \text{ if } \nu_k = \nu_1.$$

So what we have observed above is that every partition π of n in $\leq k$ Durfee squares generates $w_k(\pi)$ unrestricted partitions of n. So, we have proved

Theorem 8. *Let D_j denote the set of all partitions into exactly j successive Durfee squares. Then*

$$p(n) = \sum_{\substack{\sigma(\pi)=n \\ \pi \in D_1 \cup D_2 \cup \dots \cup D_k}} w_k(\pi).$$

Since $w_k(\pi) = 1$ for $\pi \in D_1 \cup D_2 \cup \dots \cup D_{k-1}$, it follows from Theorems 7 and 8 that

$$\sum_{\substack{\sigma(\pi)=n \\ \pi \in D_k}} w_k(\pi) = p(n) - |D_1 \cup D_2 \cup \dots \cup D_{k-1}| = p(n) - \rho_{k,k}(n). \tag{4.5}$$

5. Distinct parts and Rogers–Ramanujan partitions

In this section we obtain a weighted identity connecting $P_T(n) = Q(n)$ and $P_S(n) = \rho(n)$ (see Theorems 9 and 10 below). We first prove these identities combinatorially and then obtain their generating function form.

Consider the product

$$(-aq; q^2)_\infty(-bq^2; q^2)_\infty \tag{5.1}$$

as the generating function for partitions π into distinct parts, where π is written here as a bipartition (π_1, π_2) with π_1 having distinct odd parts and π_2 having distinct even parts. The partition (π_1, π_2) is counted with weight $a^{\nu(\pi_1)}b^{\nu(\pi_2)}$. In what follows several constructions will be described and illustrated with

$$\pi_1 : 17 + 15 + 11 + 7 + 3, \qquad \pi_2 : 20 + 18 + 14 + 8 + 6 + 2.$$

Step 1 (Decomposition). Decompose π_2 as $\pi_4 \cup \pi_5$, where π_4 consists of the parts of π_2 which are $\leq 2\nu(\pi_1)$ and π_5 has the remaining parts of π_2 (each $> 2\nu(\pi_1)$):

$$\pi_4 : 8 + 6 + 2, \qquad \pi_5 : 20 + 18 + 14.$$

Step 2. Construct $\pi_4^*(2)$, the 2–fold Ferrers conjugate of π_4, by either representing $\pi_4^*(2)$ as columns of twos or representing $\pi_4^*(2)$ as an ordinary Ferrers graph whose first and second columns are of equal length, whose third and fourth columns are of equal length, etc. In $\pi_4^*(2)$, the number of nodes in each pair of equal columns (equivalently, the sums of the twos in each column) will be the parts of π_4. Circle the two at the bottom of each column of $\pi_4^*(2)$; equivalently, circle the bottom pair of nodes in each pair of equal columns of nodes of $\pi_4^*(2)$. See Figure 4.

Step 3 (Imbedding). Consider the partition $\pi_6 = \pi_1 + \pi_4^*(2)$ whose parts are obtained by adding the number of nodes (or weights) in the corresponding rows of π_1 and $\pi_4^*(2)$:

$$\pi_6 : 23 + 19 + 15 + 9 + 3.$$

Notice that the weight of π_6 is $a^{\nu(\pi_1)}b^{\nu(\pi_4)}$. Given the partitions π_1 and $\pi_4^*(2)$, the partition $\pi_6 = \pi_1 + \pi_4^*(2)$ is uniquely defined. But given π_6, there are many

$$\pi_4^*(2): \quad \begin{array}{lll} 2 & 2 & 2 \\ 2 & 2 & \\ 2 & 2 & \\ 2 & & \end{array} \qquad\qquad \pi_4^*(2): \quad \begin{array}{llllll} \bullet & \bullet & \bullet & \bullet & \odot & \odot \\ \bullet & \bullet & \bullet & \bullet & & \\ \bullet & \bullet & \odot & \odot & & \\ \odot & \odot & & & & \end{array}$$

Figure 4

ways in which $\pi_4^*(2)$ can be extracted out of it. That is, there are several bi–partitions $(\pi_1; \pi_4^*(2))$ which will yield the same π_6. We now determine all bi–partitions $(\pi_1; \pi_4^*(2))$ which correspond to a specific π_6.

In $\pi_6 = b_1 + b_2 + \cdots + b_\nu$ with $b_1 > b_2 > \cdots > b_\nu$, the parts b_l which correspond to the circled nodes satisfy $b_\ell - b_{\ell+1} \geq 4$, where we have made the convention $b_{\nu+1} = -1$. Conversely, given a partition π_6 into distinct odd parts $b_1 > b_2 > \cdots \geq b_\nu$, consider a part b_ℓ satisfying $b_\ell - b_{\ell+1} \geq 4$, with the convention $b_{\nu+1} = -1$. From every such gap ≥ 4 we now have a choice of either extracting a pair of equal columns or not. If $b_\ell - b_{\ell+1} = 2$, we cannot extract a pair of equal columns. Thus if $b_\ell - b_{\ell+1} \geq 4$, we define the weight of b_ℓ to be $a + ab = a(1+b)$. If $b_\ell - b_{\ell+1} = 2$, we define the weight of b_ℓ to be a. Finally, the weight of the partition π_6 is the product of the weights of its parts as described above.

Step 4. Write the parts of π_5 in the form of a column in descending order and below it write the parts of π_6 in descending order to form a column C. (See the top of the next page.)

Step 5. Subtract 0 from the bottom element of C, 2 from the next one above, 4 from the one above that, and so on, and display the new values and the subtracted values in two adjacent columns $C_1|C_2$.

Step 6 (Rearrangement). Rearrange the elements of C_1 in decreasing order to form a column C_1^R.

Step 7. Form a partition π_3 by adding the corresponding elements of C_1^R and C_2. Note that π_3 is a partition with minimal difference 2. Moreover, the number of even parts of π_3 equals the number of parts of π_5, and the number of odd parts of π_3 equals $\nu(\pi_1) = \nu(\pi_6)$.

Step 4
π_5/π_6
20
18
14
23
19
15
9
3

Step 5	
C_1	C_2
6	14
6	12
4	10
15	8
13	6
11	4
7	2
3	0

Step 6	
C_1^R	C_2
15	14
13	12
11	10
7	8
6	6
6	4
4	2
3	0

Step 7
π_3
29
25
21
15
12
10
6
3

The weight of π_3 is computed as follows: All even parts of π_3 have weight b. The odd parts b_ℓ of π_3 have weight a or $a(1+b)$ depending on whether $b_\ell - b_{\ell+1} = 2$ or > 2. (Here again we have the convention that if $\pi_3 = b_1 + b_2 + \cdots + b_\nu$, then $b_{\nu+1} = -1$.) The weight $w(\pi_3)$ of π_3 is the product of the weights of its parts. With this definition of weights, the combinatorial procedure described above yields the following result:

Theorem 9. *Let D denote the set of all partitions into distinct parts. Given $\pi \in D$, let $\nu_1(\pi)$ denote the number of odd parts of π and $\nu_2(\pi)$ the number of even parts of π. Let R be the set of partitions with minimal difference 2. Then*

$$\sum_{\substack{\sigma(\pi)=n \\ \pi \in D}} a^{\nu_1(\pi)} b^{\nu_2(\pi)} = \sum_{\substack{\sigma(\pi_3)=n \\ \pi_3 \in R}} w(\pi_3).$$

Take $a = b = 1$ in Theorem 9 to get

Theorem 10. *Let $\rho_k(n)$ denote the number of Rogers–Ramanujan partitions of n in the form $n = b_1 + b_2 + \cdots + b_\nu$ such that there are exactly k odd parts b_ℓ satisfying $b_\ell - b_{\ell+1} > 2$, with the convention $b_{\nu+1} = -1$. Then*

$$Q(n) = \sum_k \rho_k(n) 2^k.$$

Remark. The combinatorial method employed above in steps 1–7 is a technique due to Bressoud but with an important variation. Bressoud [9] used the method to set up bijections between sets of partitions. For example, he used the method to give a combinatorial proof of the following result:

Göllnitz' (Little) Theorem [11]. *The number of partitions of n into parts ≥ 2 with minimal difference 2, and no consecutive odd parts, equals the number of partitions of n into distinct parts $\equiv 0, 2, 3 (mod\ 4)$.*

In Bressoud's case, all the Steps 1–7 were bijections. The main variation here is that we do not require each step to be a bijection. For example, in the proof of Theorem 9 given above, Step 3, the imbedding, is not a bijection. This greater flexibility allows us to compare partition functions which are unequal, but which become equal with weights attached.

In the above proof, partitions into distinct even parts are imbedded into partitions with distinct odd parts. One could do it the other way, namely imbed distinct odd parts into distinct even parts. In this case the transition from Step 5 to Step 6 will not be a bijection, but the imbedding (Step 3) itself will be a bijection. The method of extraction of $\pi_4^*(2)$ out of π_6 in this case will make use of a change of parity idea as discussed in our recent paper on Capparelli's conjecture (see Alladi–Andrews–Gordon [3]), where a variation in Bressoud's method is employed. In the next section another weighted identity will be discussed. There the weights arise because the transition from Step 5 to Step 6 is not a bijection.

One can obtain a generating function form of Theorem 9 by following Steps 1–7. To start with, Step 1 corresponds to the representation of the product in (5.1) in the form

$$(5.2) \qquad (-aq; q^2)_\infty (-bq^2; q^2)_\infty = \sum_{n=0}^{\infty} \frac{a^n q^{n^2}}{(q^2; q^2)_n} (-bq^2; q^2)_n (-bq^{2n+2}; q^2)_\infty.$$

At this stage it is worthwhile to introduce a third free parameter c and consider the more general series

$$\sum_{n=0}^{\infty} \frac{a^n q^{n^2}}{(q^2;q^2)_n}(-bq^2;q^2)_n(-cq^{2n+2};q^2)_\infty, \tag{5.3}$$

which is equal to

$$\begin{aligned}&\sum_{n=0}^{\infty} \frac{a_n q^{n^2}}{(q^2;q^2)_n}(-bq^2;q^2)_n \sum_{m=0}^{\infty} \frac{c^m q^{2nm} q^{m^2+m}}{(q^2;q^2)_m} \\ &= \sum_{N=0}^{\infty} \frac{q^{N^2}}{(q^2;q^2)_N} \sum_{m=0}^{N} c^m a^{N-m} q^m (-bq^2;q^2)_{N-m} \frac{(q^2;q^2)_N}{(q^2;q^2)_m (q^2;q^2)_{N-m}},\end{aligned} \tag{5.4}$$

by setting $N = m + n$. The series on the right hand side of (5.4) is the generating function for Rogers–Ramanujan partitions π_3 with weights more general than in Theorem 9, as indicated in the table below.

Let $\pi_3 : b_1 + b_2 + \cdots + b_\nu$ be a partition with minimal difference 2 and $b_{\nu+1} = -1$. Assign to each part a weight according to the following table:

Part	Condition	Weight
b_l even	–	c
b_l odd	$b_l - b_{l+1} = 2$	a
b_l odd	$b_l - b_{l+1} > 2$	$a(1+b)$

The weight $w(\pi_3)$ of π_3 is the product of weights of its parts.

The series in (5.3) is the generating function for partitions π into distinct parts counted with weights as follows. Every such partition π is written as a vector partition $\pi = (\pi_1, \pi_2, \pi_2^+)$, where π_1 has distinct odd parts, π_2 and π_2^+ have distinct even parts satisfying the conditions that $\lambda(\pi_2) \leq 2\nu(\pi_1)$ and each part of π_2^+ is $> 2\nu(\pi_1)$. The weight of π is $a^{\nu(\pi_1)} b^{\nu(\pi_2)} c^{\nu(\pi_2^+)}$. This leads to a further strengthening of Theorem 9 in the form

$$\sum_{\substack{\pi=(\pi_1,\pi_2,\pi_2^+)\\ \sigma(\pi)=n\\ \pi\in D}} a^{\nu(\pi_1)} b^{\nu(\pi_2)} c^{\nu(\pi_2^+)} = \sum_{\substack{\sigma(\pi_3)=n\\ \pi_3=R}} w(\pi_3) \tag{5.5}$$

with $w(\pi_3)$ as described by the table above.

The weighted identities derived here have several interesting applications. The first one, although very simple, is quite elegant.

Take $a = b = c = -1$. Then the series in (5.2) and (5.3) are both equal to

$$(q)_\infty = \sum_{k=-\infty}^{\infty} (-1)^k q^{(3k^2-k)/2}$$

by Euler's Pentagonal Numbers Theorem. Since $a(1+b) = 0$ in this case, we deduce from the weight table that the partitions we are counting are those consisting of consecutive odd parts starting from 1, followed by even parts each greater than every odd part. This leads to

Theorem 11. *Let $P_e(n)$ (resp. $P_0(n)$) denote the number of partitions of n into an even (resp. odd) number of distinct parts such that every even part is greater than every odd part and such that if m is an odd part, then every positive odd integer $< m$ occurs as a part. Then*

$$P_e(n) - P_0(n) = \begin{cases} (-1)^k, & \text{if} \quad n = \frac{3k^2-k}{2}, \ k = 0, \pm 1, \pm 2, \dots, \\ 0, & \text{otherwise.} \end{cases}$$

The generating function form of Theorem 11 is

$$\sum_{n=0}^{\infty} (-1)^n q^{n^2} (q^{2n+2}; q^2)_\infty = (q)_\infty = \sum_{k=-\infty}^{\infty} (-1)^k q^{(3k^2-k)/2}. \tag{5.6}$$

A more interesting application is obtained by replacing b by b/a in the weight table so that $a(1+b)$ is replaced by $a+b$. With this change, take $a=0$. So now we are counting partitions with minimal difference 2, each part ≥ 2, and no consecutive odd parts, as in Göllnitz' (little) theorem. From (5.3) we see that the generating function for such partitions is equal to

$$\begin{aligned} &\lim_{a \to 0} \sum_{n=0}^{\infty} \frac{a^n q^{n^2} \left(-\frac{b}{a} q^2; q^2\right)_n (-cq^{2n+2}; q^2)_\infty}{(q^2; q^2)_n} \\ &= \sum_{n=0}^{\infty} \frac{b^n q^{2n^2+n} (-cq^{2n+2}; q^2)_\infty}{(q^2; q^2)_n}. \end{aligned} \tag{5.7}$$

In (5.7) take $c = 1$ to get

$$(-q^2; q^2)_\infty \sum_{n=0}^{\infty} \frac{b^n q^{2n^2+n}}{(q^4; q^4)_n} = (-q^2; q^2)_\infty (-bq^3; q^4)_\infty, \tag{5.8}$$

leading to a refined form of Göllnitz' (little) theorem involving a free parameter b. This refinement is well known, but the derivation here as a limiting case $a \to 0$ is different from earlier approaches.

6. Lebesgue's identity and partitions into distinct parts

A well known identity due to Lebesgue (see Andrews [5], Ch. 2) is

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)/2} (-bq)_n}{(q)_n} = (-bq^2; q^2)_\infty (-q)_\infty. \tag{6.1}$$

By Euler's identity, the product on the right hand side of (6.1) is equal to

$$\frac{(-bq^2; q^2)_\infty}{(q; q^2)_\infty} = \sum_n \sum_k g(n,k) b^k q^n, \tag{6.2}$$

where $g(n,k)$ is the number of partitions of n in which even parts do not repeat, and there are exactly k even parts. If we take $b=1$ in (6.2), we get the generating function of

$$g(n) = \sum_k g(n;k),$$

the partition function defined in §1.

Note that Göllnitz' (little) theorem is a consequence of (6.1) under

$$q \mapsto q^2, \ b \mapsto bq^{-1}.$$

In Alladi–Gordon [4], a combinatorial proof of (6.1) was given. From the combinatorial proof the following partition theorem was deduced.

Theorem 12. *Let $g(n,k)$ be as in (6.2).*

Let $Q(n;k)$ denote the number of partitions of n into distinct parts $b_1+b_2+\cdots+b_\nu$, such that there are exactly k gaps $b_i - b_{i+1} \geq 2$, with the convention $b_{\nu+1}=0$. Then

$$\sum_k g(n,k)b^k = \sum_k Q(n,k)(1+b)^k.$$

In particular with $b=1$ in Theorem 12, we obtain the weighted identity

Theorem 13. *Let $w(\pi)=2^k$ if π is counted by $Q(n,k)$. Then*

$$g(n) = \sum_k Q(n,k)2^k = \sum_{\substack{\sigma(\pi)=n\\ \pi\in D}} w(\pi).$$

The combinatorial proof of (6.1) given in [5] considers the product in (6.1) as the generating function for bipartitions $(\pi_1;\pi_2)$, where π_1 has distinct even parts and π_2 has distinct parts. The bipartition (π_1,π_2) is counted with weight $b^{\nu(\pi_1)}$, explaining the left side of the identity in Theorem 12. The proof proceeds along Steps 1–7 as indicated in §5 by imbedding the smaller parts of π_2 into π_1 after conjugation. It turns out (see [4] for details) that every step in the proof of (6.1) is a bijection except the transition from Step 5 to Step 6. The choices here give rise to the weight $(1+b)^k$ in Theorem 12.

If we take $b=-1$ in Theorem 12, we get

$$\sum_k (-1)^k g(n,k) = Q(n,0). \tag{6.3}$$

Now $Q(n,0)$ is the number of partitions of n into distinct parts such that there are no gaps ≥ 2 in the sense described above. The only partitions without such gaps are those of the form

$$n = k + (k-1) + \cdots + 1 = k(k+1)/2.$$

Thus we get

$$Q(n,0) = \begin{cases} 1, & \text{if} \quad n = \frac{k(k+1)}{2}, \\ 0, & \text{otherwise.} \end{cases} \tag{6.4}$$

This is a combinatorial explanation of Gauss' identity

$$\sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}. \tag{6.5}$$

From (6.4) and Theorem 13 it follows that $g(n)$ is odd if and only if n is a triangular number. Thus

$$g(n) \equiv 0 (\text{mod } 2) \tag{6.6}$$

for almost all n.

More generally, the 2–adic identity in Theorem 13 contains information concerning

$$g(n) \equiv 0 (\text{mod } 2^k). \tag{6.7}$$

It is not difficult to show using Theorem 13 that

$$g(n) \equiv 0 (\text{mod } 4) \tag{6.8}$$

unless n is a sum of a triangular number and a square. Thus the congruence (6.8) holds almost always.

Actually, the exceptional set where (6.8) fails is even smaller. In a forthcoming paper [2], the validity of the congruence (6.7) for almost all n will be established for small values of k using Theorem 13 and other identities.

7. Distinct parts and powers of 2

In his classic paper of 1882, Sylvester [16] proved the following identity:

$$(-zq)_\infty = 1 + \sum_{n=1}^{\infty} \frac{z^n q^{(3n^2-n)/2}(1+zq^{2n})(-zq)_{n-1}}{(q)_n}. \tag{7.1}$$

Sylvester obtained (7.1) by analyzing partitions into distinct parts in terms of Durfee squares (see Andrews [7] for a proof of (7.1) and its relationship with other partition identities). This identity has several interesting consequences. For instance, by setting $z = -1$ in (7.1) we get

$$(q)_\infty = 1 + \sum_{n=1}^{\infty} (-1)^n q^{(3n^2-n)/2}(1+q^n) = \sum_{n=-\infty}^{\infty} (-1)^n q^{(3n^2-n)/2}, \tag{7.2}$$

which is Euler's Pentagonal Numbers Theorem.

Our purpose here is to deduce from (7.1) a new combinatorial interpretation for partitions into distinct parts that I had observed in 1990 (see Theorems 14 and 15 below).

For this purpose consider the decomposition

$$1 + zq^{2n} = 1 + zq^n - zq^n(1-q^n) \tag{7.3}$$

and rewrite (7.1) in the form

$$(-zq)_\infty = 1 + \sum_{n=1}^{\infty} \frac{z^n q^{(3n^2-n)/2}(-zq)_n}{(q)_n} - \sum_{n=1}^{\infty} \frac{z^{n+1} q^{(3n^2+n)/2}(-zq)_{n-1}}{(q)_{n-1}}. \tag{7.4}$$

The term

$$\frac{z^n q^{(3n^2-n)/2}}{(q)_n}$$

is the generating function for partitions into n parts with minimal difference 3. Therefore

$$\frac{z^n q^{(3n^2-n)/2}(-zq)_n}{(q)_n} \tag{7.5}$$

is the generating function for bipartitions $(\pi_1; \pi_2)$, where π_1 has exactly n parts with minimal difference 3, and π_2 has distinct parts $\le n$. The bipartition (π_1, π_2) is counted with weight $z^{\nu(\pi_1)+\nu(\pi_2)} = z^{n+\nu(\pi_2)}$.

Next we consider the imbedding of π_2^*, the Ferrers conjugate of π_2, into π_1, namely the partition $\pi_3 = \pi_1 + \pi_2^*$ obtained by adding the number of nodes in the corresponding rows of π_1 and π_2^*. As in the imbeddings considered above, we circle the bottom node of each column of π_2^*. Let $\pi_3 = b_1 + b_2 + \cdots + b_n$. Note that if π_2 has k parts, then π_2^* has k circled nodes. Thus π_3 has k parts corresponding to these circled nodes. Every such part b_ℓ satisfies $b_\ell - b_{\ell+1} \geq 4$ if $\ell < n$, and $b_\ell \geq 2$ if $\ell = n$. Whereas every pair $(\pi_1; \pi_2^*)$ produces a unique π_3, several bipartitions $(\pi_1; \pi_2^*)$ can be extracted out of a given π_3. In other words, given a partition π_3 with exactly k gaps ≥ 4 and least part equal to 1 or with $(k-1)$ gaps ≥ 4 and least part ≥ 2, we have a choice of either extracting or not extracting columns of π_2^* from positions where these gaps occur. Thus every partition with minimal difference 3 having either k gaps ≥ 4 and with 1 as the least part, or $k-1$ gaps ≥ 4 and least part ≥ 2, is counted with weight $z^n(1+z)^k$.

On the other hand, the term

$$\frac{q^{(3n^2+n)/2}}{(q)_{n-1}}$$

is the generating function for partitions π_1 into n parts with minimal difference 3 and with smallest part 2. Thus

$$\frac{q^{(3n^2+n)/2}(-zq)_{n-1}}{(q)_{n-1}}$$

is the generating function for bipartitions $(\pi_1; \pi_2)$ with π_1 as described above, and π_2 having distinct parts $\leq n-1$. Now by considering the imbedding of π_2^* into π_1 and its extraction we deduce that

$$\frac{z^{n+1}q^{(3n^2+n)/2}(-zq)_{n-1}}{(q)_{n-1}} \tag{7.6}$$

is the generating function for partitions $\pi_3 = b_1 + b_2 + \cdots + b_n$ into n parts with minimal difference 3, with $b_n = 2$, such that if π_3 has exactly $k-1$ gaps $b_\ell - b_{\ell+1} \geq 4$ for $1 \leq \ell < n$, then π_3 is counted with weight

$$z^{n+1}(1+z)^{k-1}.$$

Finally, observe that partitions π_3 with minimal difference 3 and with smallest part either 1 or ≥ 3 are counted by the generating function in (7.5) and not in (7.6). The partitions with minimal difference 3 and smallest part $b_n = 2$ are counted by both (7.5) and (7.6). So, the weight with which such partitions are counted on the right hand side of (7.4) is

$$z^n(1+z)^k - z^{n+1}(1+z)^{k-1} = z^n(1+z)^{k-1}.$$

Thus the series on the right in (7.4) is the generating function for partitions with minimal difference 3, such that every such partition $\pi_3 = b_1 + b_2 + \cdots + b_n$ having k gaps $b_\ell - b_{\ell+1} \geq 4$ (with the convention $b_{n+1} = -1$) is counted with weight $z^n(1+z)^k$.

We can summarize the discussion above by stating the following theorem that we just proved, which is a new interpretation of Sylvester's identity (7.1).

Theorem 14. *Let $Q_k(n)$ denote the number of partitions of n into k distinct parts.*

Let $g_3(n; \nu, k)$ denote the number of partitions of n of the form $n = b_1 + \cdots + b_\nu$ into ν parts with minimal difference 3, such that there are precisely k gaps $b_\ell - b_{\ell+1} \geq 4$, with the convention $b_{\nu+1} = -1$. Then

$$\sum_k z^k Q_k(n) = \sum_k \sum_\nu z^\nu (1+z)^k g_3(n; \nu, k).$$

In Theorem 14 take $z = 1$ to get

Theorem 15. *Let*

$$g_3(n,k) = \sum_{\nu} g_3(n;\nu,k).$$

Then

$$Q(n) = \sum_{k} g_3(n;k)2^k.$$

We believe that Theorem 15 is the key to the combinatorial understanding of the behavior of $Q(n)(\text{mod } 2^k)$, just as Theorem 13 of §6 can be used to study $g(n)(\text{mod } 2^k)$.

It is well known in the theory of modular forms (see Serre [15]) that the coefficients c_n of certain modular forms satisfy

$$c_n = 0(\text{mod } m)$$

for almost all n, for any integer m. Thus one way of proving that, for a given k,

$$Q(n) \equiv 0(\text{mod } 2^k) \tag{7.7}$$

for almost all n, is to construct such a modular form

$$f_k(z) = \sum_{n=1}^{\infty} c_n e^{2\pi i n\tau}$$

with the coefficients satisfying the condition

$$c_n \equiv Q(n)(\text{mod } 2^k).$$

The difficulty is that there is no single modular form $f(\tau)$ which would work for all k. In fact an infinite family of such modular forms needs to be constructed. It was only a few years ago that Gordon provided such a construction and thus proved the striking result that for each positive integer k, the congruence (7.7) is valid for almost all n. In a forthcoming paper [13], Gordon and Ono generalize this result to $p-$regular partitions as follows.

A $p-$regular partition is one whose parts repeat $< p$ times. Thus the generating function for $Q^{(p)}(n)$, the number of $p-$regular partitions of n, is

$$\sum_{n=0}^{\infty} Q^{(p)}(n)q^n = \prod_{j=1}^{\infty}(1+q^j+q^{2j}+\cdots+q^{(p-1)j}) = \frac{(q)_\infty}{(q^p;q^p)_\infty}.$$

Making heavy use of the theory of modular forms, Gordon and Ono [13] prove

Theorem 16. *For every positive integer k, the congruence*

$$Q^{(p)}(n) \equiv 0(\mathit{mod}\ p^k)$$

holds for almost all n.

Since $Q(n)$ is a fundamental function in the theory of partitions, it is obviously of interest to provide a combinatorial explanation for the validity of the congruence (7.7) almost always. We believe that Theorem 15 contains such combinatorial information, and in 1990 we used it study (7.7) for $k \le 4$.

For example, to determine $Q(n)(\text{mod } 2)$ we need only consider the term $g_3(n,0)$. Thus $Q(n)$ is odd if and only if $g_3(n,0)$ is odd. Now $g_3(n,0)$ is the number of partitions of n in the form $b_1+b_2+\cdots+b_\nu$ such that the minimal difference is 3, and that there are no gaps $b_i - b_{i+1} \ge 4$ with the convention $b_{\nu+1} = -1$. So this leaves us with only two choices.

Case 1. $b_\nu = 1$, $b_i - b_{i+1} = 3$, $1 \le i < \nu$.

Case 2. $b_\nu = 2$, $b_i - b_{i+1} = 3$, $1 \le i < \nu$.

In Case 1 the integer n being partitioned is $(3\nu^2 - \nu)/2$; in Case 2 the number being partitioned is $(3\nu^2 + \nu)/2$. Thus $Q(n)$ is odd if and only if n is a pentagonal number.

By analyzing the 2–adic identity in Theorem 15 more closely, it can be shown (see [2] for details) that

$$Q(n) \equiv 0 (\bmod 4)$$

unless $n = P_k + j^2$ can be written as a pentagonal number plus a perfect square. Thus the exceptional set has density zero. Actually, the exceptional set is even thinner than the one given above, as is shown in [2].

As k gets larger, the determination of the exceptional sets using Theorem 15 becomes more and more complicated, and in [2] we investigate the cases $k \le 4$. Involved in the study are the properties of various arithmetical functions like the divisor function.

Theorem 15 has many other implications. For instance, it can be refined by introducing three free parameters, as is shown in [1]. It is such a refinement that leads to a combinatorial correspondence related to a very deep theorem of Göllnitz [11]. Another consequence of this three parameter refinement is a new interpretation for Jacobi's triple product identity. In view of the importance of this refinement, a detailed discussion of this is done separately [1].

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References

1. K. Alladi, *A combinatorial correspondence related to Göllnitz' (big) partition theorem and applications*, Trans. Amer. Math. Soc. (to appear).
2. K. Alladi, *On the divisibility of certain partition functions by small powers of 2* (in preparation).
3. K. Alladi, G. E. Andrews and B. Gordon, *Refinements and generalizations of Capparelli's conjecture on partitions*, J. Algebra **174** (1995), 636-658. MR **96b:**11136
4. K. Alladi and B. Gordon, *Partition identities and a continued fraction of Ramanujan*, J. Comb. Th. Ser. A **63** (1993), 275-300. MR **94f:**11108
5. G. E. Andrews, *The theory of partitions,* Encyclopedia of Mathematics and its Applications, Vol. 2, Addison-Wesley, Reading (1976). MR **58:**27738
6. G. E. Andrews, *Partitions and Durfee dissection*, Amer. J. Math. **10** (1979), 735-742. MR **80h:**10020
7. G. E. Andrews, *Partitions: Yesterday and Today*, New Zealand Math. Soc., Wellington (1979). MR **81b:**01015
8. D. Bowman, *Partitions with numbers in their gaps*, Acta Arith. **74** (1996), 97-105. MR **96m:**11090
9. D. M. Bressoud, *On a partition theorem of Göllnitz*, J. Reine Angew. Math. **305** (1979), 215-217. MR **80a:**10027
10. F. Garvan, *Generalizations of Dyson's rank and non–Rogers–Ramanujan partitions*, Manus. Math **84** (1994), 343-359. MR **95e:**11115

11. H. Göllnitz, *Partitionen with Differenzenbedingungen*, J. Reine Angew. Math **225** (1967), 154-190. MR **35:**2848
12. B. Gordon, *A combinatorial generalization of the Rogers–Ramanujan identities*, Amer. J. Math. **83** (1961), 393-399. MR **23:**A809
13. B. Gordon and K. Ono, *Divisibility of certain partition functions by powers of primes*, The Ramanujan Journal **1** (1997), 25-34.
14. G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, 4th ed., Clarendon Press, Oxford (1960). MR **81i:**10002 (5th ed.)
15. J. P. Serre, *Divisibilité des coefficients des formes modulaires de poids entiers*, C. R. Acad. Sci. Paris A **279** (1974), 679-682. MR **52:**3060
16. J. J. Sylvester, *A constructive theory of partitions arranged in three Acts, an Interact and an Exodion*, Amer. J. Math. **5** (1882), 251-330.

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